

Solutions to Maths workbook – 1 | Trigonometric Identities and Equations

Level - 3

Daily Tutorial Sheet - 15

168. Let $\theta = \frac{(2n+1)\pi}{7}$, where $n = 0, 1, 2, 3, 4, 5$, then, $4\theta = (2n+1)\pi - 3\theta$

$$\Rightarrow \cos 4\theta = -\cos 3\theta \Rightarrow 2\cos^2 2\theta - 1 = -(4\cos^3 \theta - 3\cos \theta)$$

$$\Rightarrow 2(\cos^2 \theta - 1)^2 - 1 = -4\cos^3 \theta + 3\cos \theta \Rightarrow 2(2x^2 - 1)^2 - 1 = -4x^3 + 3x \quad [\text{Put } x = \cos \theta]$$

$$\Rightarrow 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0 \Rightarrow (x+1)(8x^3 - 4x^2 - 4x + 1) = 0$$

The roots of this equation are, $\cos \frac{\pi}{7}, \cos \frac{3\pi}{7}, \cos \frac{5\pi}{7}, \cos \frac{7\pi}{7}, \cos \frac{9\pi}{7}, \cos \frac{11\pi}{7}$

$\therefore$ The roots of $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos \frac{\pi}{7}, \cos \frac{3\pi}{7}, \cos \frac{5\pi}{7}$ (i)

Put $x = \frac{1}{y}$ in equation (i) (i.e. $y = \sec \theta$), then $\sec \frac{\pi}{7}, \sec \frac{3\pi}{7}, \sec \frac{5\pi}{7}$ are the roots of the equation.

$$\frac{8}{y^3} - \frac{4}{y^2} - \frac{4}{y} + 1 = 0 \Rightarrow y^3 - 4y^2 - 4y + 8 = 0 \Rightarrow \sec \frac{\pi}{7} + \sec \frac{3\pi}{7} + \sec \frac{5\pi}{7} = 4$$

Again putting $\frac{1}{x^2} = y$ in equation (i) (i.e. $y = \sec^2 \theta$)

$$\frac{8}{y^{3/2}} - \frac{4}{y} - \frac{4}{y^{1/2}} + 1 = 0 \Rightarrow 8 - 4y^{1/2} - 4y + y^{3/2} = 0 \Rightarrow y^{1/2}(y - 4) = 4(y - 2)$$

$$\Rightarrow y(y - 4)^2 = 16(y - 2)^2 \Rightarrow y^3 - 24y^2 + 80y - 64 = 0$$

Hence, the roots are $\sec^2 \frac{\pi}{7}, \sec^2 \frac{3\pi}{7}, \sec^2 \frac{5\pi}{7}$

Now, putting $y = 1 + z$ (i.e. $z = \tan^2 \theta$) we have, $(1 + z)^3 - 24(1 + z)^2 + 80(1 + z) - 64 = 0$

$$\Rightarrow z^3 - 21z^2 + 35z - 7 = 0 \text{ whose roots are } \tan^2 \frac{\pi}{7}, \tan^2 \frac{3\pi}{7}, \tan^2 \frac{5\pi}{7}$$

169. Here, $3\cos 2\theta = 1 \Rightarrow 3(2\cos^2 \theta - 1) = 1 \Rightarrow 6\cos^2 \theta = 4; \Rightarrow \cos^2 \theta = \frac{2}{3}$

$$\text{Now, } \tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1 - \cos^2 \theta}{\frac{2}{3}} = \frac{1 - \frac{2}{3}}{\frac{2}{3}} = \frac{1}{2} \Rightarrow 32 \tan^8 \theta = 2\cos^2 \alpha - 3\cos \alpha$$

$$\Rightarrow 32\left(\frac{1}{2}\right)^4 = 2\cos^2 \alpha - 3\cos \alpha \Rightarrow 2\cos^2 \alpha - 3\cos \alpha - 2 = 0$$

$$\Rightarrow (2\cos \alpha + 1)(\cos \alpha - 2) = 0 \Rightarrow 2\cos \alpha + 1 = 0 \text{ or } \cos \alpha - 2 = 0$$

$$\Rightarrow \cos \alpha = -\frac{1}{2} \text{ or } \cos \alpha = 2 \text{ (impossible)} \Rightarrow \alpha = 2n\pi \pm \left(\pi - \frac{\pi}{3}\right) = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$$

170. The given equation is, $\sin y = 2\left(\sin^4 x + \frac{1}{\sin^4 x} + 2\right) + 2\left(\cos^4 x + \frac{1}{\cos^4 x} + 2\right) - 24$

$$= 2(\sin^4 x + \cos^4 x) + 2\left(\frac{1}{\sin^4 x} + \frac{1}{\cos^4 x}\right) + 8 - 24 = 2(\sin^4 x + \cos^4 x)\left\{1 + \frac{1}{\sin^4 x \cdot \cos^4 x}\right\} - 16$$

$$\Rightarrow \sin y = 2\left(1 - \frac{1}{2}\sin^2 2x\right)\left(1 + \frac{16}{\sin^4 2x}\right) - 16 \Rightarrow 16 + \sin y = (2 - \sin^2 2x)\left(1 + \frac{16}{\sin^4 2x}\right) \dots (i)$$

Since, $\sin y \leq 1 \Rightarrow 16 + \sin y \leq 17$

$\therefore$ L.H.S. is not greater than 17, on the other hand $0 \leq \sin^2 2x \leq 1$

$$\Rightarrow 2 - \sin^2 2x \geq 1 \text{ and } \sin^4 2x \leq 1 \Rightarrow \frac{16}{\sin^4 2x} \geq 16 \Rightarrow 1 + \frac{16}{\sin^4 2x} \geq 17$$

$$\Rightarrow (2 - \sin^2 2x)\left(1 + \frac{16}{\sin^4 2x}\right) \geq 17$$

This shows that right member of equation (i) is not less than 17.

Thus the inequality holds only when $\sin y = 1$ and $\sin^2 2x = 1$

$$\Rightarrow y = (4n+1)\frac{\pi}{2}, n \in I \text{ and } \Rightarrow 2x = m\pi \pm \frac{\pi}{2}, m \in I \Rightarrow x = (2m \pm 1)\frac{\pi}{4}$$

Thus, the solutions are $\left((2m \pm 1)\frac{\pi}{4}, (4n+1)\frac{\pi}{2}\right)$

171. We have, $4\left[3\sqrt{4x-x^2}\left(\frac{1-\cos(x+y)}{2}\right) + 2\cos(x+y)\right] = 13 + 4\cos^2(x+y)$

$$\Rightarrow 6\sqrt{4x-x^2} - 6\sqrt{4x-x^2}\cos(x+y) + 8\cos(x+y) = 13 + 4\cos^2(x+y)$$

$$\Rightarrow 4\cos^2(x+y) + (6\sqrt{4x-x^2} - 8)\cos(x+y) + (13 - 6\sqrt{4x-x^2}) = 0 \dots (i)$$

$$\text{Let, } \cos(x+y) = t \text{ and } 6\sqrt{4x-x^2} - 8 = a \Rightarrow 6\sqrt{4x-x^2} = 8 + a$$

$$\text{Clearly, } 4x - x^2 \geq 0 \text{ or } x(x-4) \leq 0 \text{ or } 0 \leq x \leq 4 \dots (ii)$$

Now, equation (i) reduces to ; $4t^2 + at + (5-a) = 0$ which is quadratic in t .

$$D \geq 0 \Rightarrow a^2 - 4.4(5-a) \geq 0 \Rightarrow a^2 + 16a - 80 \geq 0 \Rightarrow (a+20)(a-4) \geq 0 \Rightarrow a \leq -20 \text{ or } a \geq 4 \dots (iii)$$

$$\text{However, according to substitution; } a = 6\sqrt{4x-x^2} - 8 = 6\sqrt{4-(x-2)^2} - 8 \Rightarrow -8 \leq a \leq 4$$

$\dots (iv)$

$$\Rightarrow a = 4 \quad \{\text{from (iii) and (iv)}\}$$

$$\text{Hence, } 6\sqrt{4x-x^2} - 8 = 4 \Rightarrow \sqrt{4x-x^2} = 2 \Rightarrow x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \text{ or } x = 2$$

$$\text{Now, equation (i) becomes : } 4\cos^2(2+y) + 4\cos(2+y) + 1 = 0 \Rightarrow (2\cos(2+y) + 1)^2 = 0$$

$$\Rightarrow 2\cos(2+y) + 1 = 0$$

$$\Rightarrow \cos(2+y) = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) \Rightarrow 2+y = 2n\pi \pm \frac{2\pi}{3} \Rightarrow y = 2n\pi \pm \frac{2\pi}{3} - 2, n \in I$$

Thus, the solutions are $\left(2, 2n\pi \pm \frac{2\pi}{3} - 2\right)$

172. Rewriting the given equation as $\{\tan(x+y) - \cot(x+y)\}^2 = -(1+2x+x^2) = -(1+x)^2$

or $\{\tan(x+y) - \cot(x+y)\}^2 + (1+x)^2 = 0$ which is possible only when:

$$\tan(x+y) - \cot(x+y) = 0 \text{ and } 1+x = 0 \Rightarrow \tan^2(x+y) = 1 = \tan^2 \frac{\pi}{4} \text{ and } x = -1$$

$$\text{Now, } \tan^2(x+y) = \tan^2 \frac{\pi}{4} \Rightarrow x+y = n\pi \pm \frac{\pi}{4} \Rightarrow y = n\pi \pm \frac{\pi}{4} + 1, n \in I$$

∴ The required solutions are $\left(-1, n\pi \pm \frac{\pi}{4} + 1\right), n \in I$.

173. We know, $a + b + c = 0 \Leftrightarrow a^3 + b^3 + c^3 = 3abc$, Put $a = \sin x, b = \cos x, c = 1$

$$= 1 + \sin^3 x + \cos^3 x + 3 \sin x \cdot \cos x = \frac{3}{2} \sin 2x \quad \text{as } a + b + c = 0 \Rightarrow 1 + \sin x + \cos x = 0$$

$$\Rightarrow \sqrt{2} \cos\left(x - \frac{\pi}{4}\right) = -1 \Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos \frac{3\pi}{4} \Rightarrow x = 2n\pi \pm \frac{3\pi}{4} + \frac{\pi}{4}, n \in I.$$